

Portrait of Isaac Newton at about the age of sixty, presented by Newton to David Gregory (1661–1708). This small oval drawing (roughly $3\frac{3}{4}$ in. from top to bottom and $3\frac{1}{4}$ in. from left to right) is closely related to the large oval portrait in oils made by Kneller in 1702, which is considered to be the second authentic portrait made of Newton. The kinship between this drawing and the oil painting can be seen in the pose, the expression, and such unmistakable details as the slight cast in the left eye and the button on the shirt. Newton is shown in both this drawing and the painting of 1702 in his academic robe and wearing a luxurious wig, whereas in the previous portrait by Kneller (now in the National Portrait Gallery in London), painted in 1689, two years after the publication of the *Principia*, Newton is similarly attired but is shown with his own shoulder-length hair.

This drawing was almost certainly made after the painting, since Kneller's preliminary drawings for his paintings are usually larger than this one and tend to concentrate on the face rather than on the details of the attire of the subject. The fact that this drawing shows every detail of the finished oil painting is thus evidence that it was copied from the finished portrait. Since Gregory died in 1708, the drawing can readily be dated to between 1702 and 1708. In those days miniature portraits were commonly used in the way that we today would use portrait photographs. The small size of the drawing indicates that it was not a copy made in preparation for an engraved portrait but was rather made to be used by Newton as a gift.

The drawing captures Kneller's powerful representation of Newton, showing him as a person with a forceful personality, poised to conquer new worlds in his recently gained position of power in London. This high level of artistic representation and the quality of the drawing indicate that the artist responsible for it was a person of real talent and skill.

The drawing is mounted in a frame, on the back of which there is a longhand note reading: "This original drawing of Sir Isaac Newton, belonged formerly to Professor Gregory of Oxford; by him it was bequeathed to his youngest son (Sir Isaac's godson) who was later Secretary of Sion College; & by him left by Will to the Revd. Mr. Mence, who had the Goodness to give it to Dr. Douglas; March 8th 1870."

David Gregory first made contact with Newton in the early 1690s, and although their relations got off to a bad start, Newton did recommend Gregory for the Savilian Professorship of Astronomy at Oxford, a post which he occupied until his death in 1708. As will be evident to readers of the Guide, Gregory is one of our chief sources of information concerning Newton's intellectual activities during the 1690s and the early years of the eighteenth century, the period when Newton was engaged in revising and planning a reconstruction of his *Principia*. Gregory recorded many conversations with Newton in which Newton discussed his proposed revisions of the *Principia* and other projects and revealed some of his most intimate and fundamental thoughts about science, religion, and philosophy. So far as is known, the note on the back of the portrait is the only record that Newton stood godfather to Gregory's youngest son.

ISAAC NEWTON



THE *PRINCIPIA*

Mathematical Principles of Natural Philosophy



A New Translation

by I. Bernard Cohen and Anne Whitman

assisted by Julia Budenz

Preceded by

A GUIDE TO NEWTON'S *PRINCIPIA*

by I. Bernard Cohen



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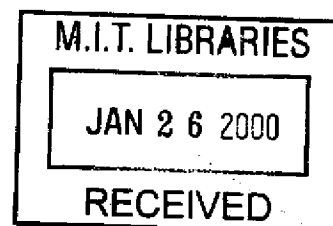
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*This translation is
dedicated to*

D. T. WHITESIDE

with respect and affection



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A GUIDE TO NEWTON'S *PRINCIPIA*



By I. Bernard Cohen

with contributions by Michael Nauenberg (§3.9)

and George E. Smith (§§7.10, 8.8, 8.15, 8.16, and 10.19)

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Error and doubt no longer encumber us with mist;
 For the keenness of a sublime intelligence has made it possible for us to enter
 The dwellings of the gods above and to climb the heights of heaven.

Mortals arise, put aside earthly cares,
 And from this treatise discern the power of a mind sprung from heaven,
 Far removed from the life of beasts.
 He who commanded us by written tablets to abstain from murder,
 Thefts, adultery, and the crime of bearing false witness,
 Or he who taught nomadic peoples to build walled cities, or he who enriched the
 nations with the gift of Ceres,
 Or he who pressed from the grape a solace for cares,
 Or he who with a reed from the Nile showed how to join together
 Pictured sounds and to set spoken words before the eyes,
 Exalted the human lot less, inasmuch as he was concerned with only a few
 comforts of a wretched life,
 And thus did less than our author for the condition of mankind.
 But we are now admitted to the banquets of the gods;
 We may deal with the laws of heaven above; and we now have
 The secret keys to unlock the obscure earth; and we know the immovable order
 of the world
 And the things that were concealed from the generations of the past.

O you who rejoice in feeding on the nectar of the gods in heaven,
 Join me in singing the praises of NEWTON, who reveals all this,
 Who opens the treasure chest of hidden truth,
 NEWTON, dear to the Muses,
 The one in whose pure heart Phoebus Apollo dwells and whose mind he has filled
 with all his divine power;
 No closer to the gods can any mortal rise.

Edm. Halley

Author's Preface to the Reader

SINCE THE ANCIENTS (according to Pappus) considered *mechanics* to be of the greatest importance in the investigation of nature and science and since the moderns—rejecting substantial forms and occult qualities—have undertaken to reduce the phenomena of nature to mathematical laws, it has seemed best in this treatise to concentrate on *mathematics* as it relates to natural philosophy. The ancients divided *mechanics* into two parts: the *rational*, which proceeds rigorously through demonstrations, and the *practical*.^a *Practical mechanics* is the subject that comprises all the manual arts, from which the subject of *mechanics* as a whole has adopted its name. But since those who practice an art do not generally work with a high degree of exactness, the whole subject of *mechanics* is distinguished from *geometry* by the attribution of exactness to *geometry* and of anything less than exactness to *mechanics*. Yet the errors do not come from the art but from those who practice the art. Anyone who works with less exactness is a more imperfect mechanic, and if anyone could work with the greatest exactness, he would be the most perfect mechanic of all. For the description of straight lines and circles, which is the foundation of *geometry*, appertains to *mechanics*. *Geometry*

All notes to the translation are keyed to the text by superscript letters. When a note is introduced by two letters, such as "aa," it refers to that part of the text enclosed between an opening superscript "a" and a final or closing "a."

These notes are, for the most part, extracts from variant passages or expressions as found in the first two editions. The glosses and explanations of the text are to be found in the Guide, the text of which follows the order of Newton's presentation in the *Principia*.

a. Newton's comparison and contrast between the subject of rational or theoretical mechanics and practical mechanics was a common one at the time of the *Principia*. Thus John Harris in his *Newtonian Lexicon Technicum* (London, 1704), citing the authority of John Wallis, made a distinction between the two as follows. One was a "Geometry of Motion," a "Mathematical Science which shews the Effects of Powers, or moving Forces," and "demonstrates the Laws of Motion." The other is "commonly taken for those Handy-crafts, which require as well the Labour of the Hands, as the Study of the Brain." The subject of the *Principia* became generally known as "rational mechanics" following Newton's use of that name in his Preface.

does not teach how to describe these straight lines and circles, but postulates such a description. For *geometry* postulates that a beginner has learned to describe lines and circles exactly before he approaches the threshold of *geometry*, and then it teaches how problems are solved by these operations. To describe straight lines and to describe circles are problems, but not problems in *geometry*. *Geometry* postulates the solution of these problems from *mechanics* and teaches the use of the problems thus solved. And *geometry* can boast that with so few principles obtained from other fields, it can do so much. Therefore *geometry* is founded on mechanical practice and is nothing other than that part of *universal mechanics* which reduces the art of measuring to exact propositions and demonstrations. But since the manual arts are applied especially to making bodies move, *geometry* is commonly used in reference to magnitude, and *mechanics* in reference to motion. In this sense *rational mechanics* will be the science, expressed in exact propositions and demonstrations, of the motions that result from any forces whatever and of the forces that are required for any motions whatever. The ancients studied this part of *mechanics* in terms of the *five powers* that relate to the manual arts [i.e., the five mechanical powers] and paid hardly any attention to gravity (since it is not a manual power) except in the moving of weights by these powers. But since we are concerned with natural philosophy rather than manual arts, and are writing about natural rather than manual powers, we concentrate on aspects of gravity, levity, elastic forces, resistance of fluids, and forces of this sort, whether attractive or impulsive. And therefore our present work sets forth mathematical principles of natural philosophy. For the basic problem [*lit.* whole difficulty^b] of philosophy seems to be to discover the forces of nature from the phenomena of motions and then to demonstrate the other phenomena from these forces. It is to these ends that the general propositions in books 1 and 2 are directed, while in book 3 our explanation of the system of the world illustrates these propositions. For in book 3, by means of propositions demonstrated mathematically in books 1 and 2, we derive from celestial phenomena the gravitational forces by which bodies tend toward the sun and toward the individual planets. Then the motions of the planets, the comets, the moon, and the sea are deduced from these forces by propositions that are also mathematical. If only we could derive the other phenomena of nature from mechanical principles by the same kind of reasoning! For many things lead me to have a suspicion that all phenomena may depend on certain forces by which the particles of bodies, by causes not yet known, either are impelled toward one another and cohere in regular figures, or are repelled

b. Newton would seem to be expressing in Latin more or less the same concept that later appears in English (in query 28 of the *Opticks*) as "the main Business of natural Philosophy."

from one another and recede. Since these forces are unknown, philosophers have hitherto made trial of nature in vain. But I hope that the principles set down here will shed some light on either this mode of philosophizing or some truer one.

In the publication of this work, Edmond Halley, a man of the greatest intelligence and of universal learning, was of tremendous assistance; not only did he correct the typographical errors and see to the making of the woodcuts, but it was he who started me off on the road to this publication. For when he had obtained my demonstration of the shape of the celestial orbits, he never stopped asking me to communicate it to the Royal Society, whose subsequent encouragement and kind patronage made me begin to think about publishing it. But after I began to work on the inequalities of the motions of the moon, and then also began to explore other aspects of the laws and measures of gravity and of other forces, the curves that must be described by bodies attracted according to any given laws, the motions of several bodies with respect to one another, the motions of bodies in resisting mediums, the forces and densities and motions of mediums, the orbits of comets, and so forth, I thought that publication should be put off to another time, so that I might investigate these other things and publish all my results together. I have grouped them together in the corollaries of prop. 66 the inquiries (which are imperfect) into lunar motions, so that I might not have to deal with these things one by one in propositions and demonstrations, using a method more prolix than the subject warrants, which would have interrupted the sequence of the remaining propositions. There are a number of things that I found afterward which I preferred to insert in less suitable places rather than to change the numbering of the propositions and the cross-references. I earnestly ask that everything be read with an open mind and that the defects in a subject so difficult may be not so much reprehended as investigated, and kindly supplemented, by new endeavors of my readers.

Trinity College, Cambridge
8 May 1686

Is. Newton

DEFINITIONS



^a*Quantity of matter is a measure of matter that arises from its density and volume jointly.*^a Definition 1

^bIf the density of air is doubled in a space that is also doubled, there is four times as much air, and there is six times as much if the space is tripled.^b The case is the same for snow and powders condensed by compression or liquefaction, and also for all bodies that are condensed in various ways by any causes whatsoever. For the present, I am not taking into account any medium, if there should be any, freely pervading the interstices between the parts of

aa. In translating def. 1, we have rendered Newton's "*Quantitas materiae est mensura ejusdem...*" as "*Quantity of matter is a measure of matter...*" rather than the customary "... is the measure..." The indefinite article is more in keeping with the Latin usage, with its absence of articles, and accords better with the sense in which we have interpreted this definition. See the Guide, §4.2. It should be noted that the indefinite article permits the possibility of the sense of either a definite or an indefinite article, whereas a definite article precludes the possibility of the sense of an indefinite article.

bb. Ed. 3 reads literally: "Air, if the density is doubled, in a space also doubled, becomes quadruple; in [a space] tripled, sextuple." The printer's manuscript for ed. 1 and the printed text of ed. 1 have: "Air twice as dense in twice the space is quadruple." Newton's interleaved copy of ed. 1 has: "Air twice as dense in twice the space is quadruple; in three times [the space], sextuple." Newton's annotated copy of ed. 1 has first: "Air twice as dense in twice the space becomes quadruple; in three times [the space], sextuple." This is then deleted and replaced with: "Air, by doubling the density, in the same container becomes double; in a container twice as large, quadruple; in one three times as large, sextuple; and by tripling the density, it becomes triple in the same container and sextuple in a container twice as large," but the last clause, "and by tripling... large," is then deleted.

The manuscript errata at the end of the annotated copy have: "For this quantity, if the density is given [or fixed], is as the volume and, if the volume is given, is as the density and therefore, if neither is given, is as the product of both. Thus indeed Air, if the density is doubled, in a space also doubled, becomes quadruple; in [a space] tripled, sextuple." The first sentence, "For this... product of both," and the following two words, "Thus indeed," are inserted over a caret preceding "Air."

An interleaf of the interleaved copy of ed. 1 and then the printed text of ed. 2 have exactly the same phrasing as ed. 3.

bodies. Furthermore, I mean this quantity whenever I use the term "body" or "mass" in the following pages. It can always be known from a body's weight, for—by making very accurate experiments with pendulums—I have found it to be proportional to the weight, as will be shown below.

Definition 2 *Quantity of motion is a measure of motion that arises from the velocity and the quantity of matter jointly.*

The motion of a whole is the sum of the motions of the individual parts, and thus if a body is twice as large as another and has equal velocity there is twice as much motion, and if it has twice the velocity there is four times as much motion.

Definition 3 *Inherent force of matter is the power of resisting by which every body, ^aso far as it is able,^a perseveres in its state either of resting or of moving ^buniformly straight forward.^b*

This force is always proportional to the body and does not differ in any way from the inertia of the mass except in the manner in which it is conceived. Because of the inertia of matter, every body is only with difficulty put out of its state either of resting or of moving. Consequently, inherent force may also be called by the very significant name of force of inertia.^c Moreover, a body exerts this force only during a change of its state, caused by another force impressed upon it, and this exercise of force is, depending on the viewpoint, both resistance and impetus: resistance insofar as the body, in order to maintain its state, strives against the impressed force, and impetus insofar as the same body, yielding only with difficulty to the force of a resisting obstacle, endeavors to change the state of that obstacle. Resistance is commonly attributed to resting bodies and impetus to moving bodies; but

aa. Newton's Latin clause is "quantum in se est," which here means "to the degree that it can of and by itself." See I. Bernard Cohen, "'Quantum in se est': Newton's Concept of Inertia in Relation to Descartes and Lucretius," *Notes and Records of the Royal Society of London* 19 (1964): 131–155.

bb. Newton's "in directum" (used together with "uniformiter" ["uniformly"]) has the sense of moving straight on, of going continuously straight forward, and therefore in a straight line. In an earlier version, Newton had used the phrase "in linea recta" ("in a right line" or "in a straight line"), but by the time of the *Principia* he had rejected this expression in favor of "in directum." For details, see the Guide, §10.2. On Newton's "vis insita" and our rendition, see the Guide, §4.7.

c. Newton's interleaved copy of ed. 2 adds the following, which was never printed: "I do not mean Kepler's force of inertia, by which bodies tend toward rest, but a force of remaining in the same state either of resting or of moving."

motion and rest, in the popular sense of the terms, are distinguished from each other only by point of view, and bodies commonly regarded as being at rest are not always truly at rest.

Impressed force is the action exerted on a body to change its state either of resting or of moving uniformly straight forward. **Definition 4**

This force consists solely in the action and does not remain in a body after the action has ceased. For a body perseveres in any new state solely by the force of inertia. Moreover, there are various sources of impressed force, such as percussion, pressure, or centripetal force.

Centripetal force is the force by which bodies are drawn from all sides, are impelled, or in any way tend, toward some point as to a center. **Definition 5**

One force of this kind is gravity, by which bodies tend toward the center of the earth; another is magnetic force, by which iron seeks a lodestone; and yet another is that force, whatever it may be, by which the planets are continually drawn back from rectilinear motions and compelled to revolve in curved lines.

^aA stone whirled in a sling endeavors to leave the hand that is whirling it, and by its endeavor it stretches the sling, doing so the more strongly the more swiftly it revolves; and as soon as it is released, it flies away. The force opposed to that endeavor, that is, the force by which the sling continually draws the stone back toward the hand and keeps it in an orbit, I call centripetal, since it is directed toward the hand as toward the center of an orbit. And the same applies to all bodies ^bthat are made to move in orbits.^b They all endeavor to recede from the centers of their orbits, and unless some force opposed to that endeavor is present, restraining them and keeping them in orbits and hence called by me centripetal, they will go off in straight lines with uniform motion. If a projectile were deprived of the force of gravity, it would not be deflected toward the earth but would go off in a straight line into the heavens and do so with uniform motion, provided that the resistance of the air were removed. The projectile, by its gravity, is drawn back from a rectilinear course and continually deflected toward the earth, and this is so

aa. Ed. 1 lacks this.

bb. See the Guide, §2.4.

to a greater or lesser degree in proportion to its gravity and its velocity of motion. The less its gravity in proportion to its quantity of matter, or the greater the velocity with which it is projected, the less it will deviate from a rectilinear course and the farther it will go. If a lead ball were projected with a given velocity along a horizontal line from the top of some mountain by the force of gunpowder and went in a curved line for a distance of two miles before falling to the earth, then the same ball projected with twice the velocity would go about twice as far and with ten times the velocity about ten times as far, provided that the resistance of the air were removed. And by increasing the velocity, the distance to which it would be projected could be increased at will and the curvature of the line that it would describe could be decreased, in such a way that it would finally fall at a distance of 10 or 30 or 90 degrees or even go around the whole earth or, lastly, go off into the heavens and continue indefinitely in this motion. And in the same way that a projectile could, by the force of gravity, be deflected into an orbit and go around the whole earth, so too the moon, whether by the force of gravity—if it has gravity—or by any other force by which it may be urged toward the earth, can always be drawn back toward the earth from a rectilinear course and deflected into its orbit; and without such a force the moon cannot be kept in its orbit. If this force were too small, it would not deflect the moon sufficiently from a rectilinear course; if it were too great, it would deflect the moon excessively and draw it down from its orbit toward the earth. In fact, it must be of just the right magnitude, and mathematicians have the task of finding the force by which a body can be kept exactly in any given orbit with a given velocity and, alternatively, to find the curvilinear path into which a body leaving any given place with a given velocity is deflected by a given force.^a

The quantity of centripetal force is of three kinds: absolute, accelerative, and motive.

Definition 6 *The absolute quantity of centripetal force is the measure of this force that is greater or less in proportion to the efficacy of the cause propagating it from a center through the surrounding regions.*

An example is magnetic force, which is greater in one lodestone and less in another, in proportion to the bulk or potency of the lodestone.

The accelerative quantity of centripetal force is the measure of this force that is proportional to the velocity which it generates in a given time. **Definition 7**

One example is the potency of a lodestone, which, for a given lodestone, is greater at a smaller distance and less at a greater distance. Another example is the force that produces gravity, which is greater in valleys and less on the peaks of high mountains and still less (as will be made clear below) at greater distances from the body of the earth, but which is everywhere the same at equal distances, because it equally accelerates all falling bodies (heavy or light, great or small), provided that the resistance of the air is removed.

The motive quantity of centripetal force is the measure of this force that is proportional to the motion which it generates in a given time. **Definition 8**

An example is weight, which is greater in a larger body and less in a smaller body; and in one and the same body is greater near the earth and less out in the heavens. This quantity is the centripetency, or propensity toward a center, of the whole body, and (so to speak) its weight, and it may always be known from the force opposite and equal to it, which can prevent the body from falling.

These quantities of forces, for the sake of brevity, may be called motive, accelerative, and absolute forces, and, for the sake of differentiation, may be referred to bodies seeking a center, to the places of the bodies, and to the center of the forces: that is, motive force may be referred to a body as an endeavor of the whole directed toward a center and compounded of the endeavors of all the parts; accelerative force, to the place of the body as a certain efficacy diffused from the center through each of the surrounding places in order to move the bodies that are in those places; and absolute force, to the center as having some cause without which the motive forces are not propagated through the surrounding regions, whether this cause is some central body (such as a lodestone in the center of a magnetic force or the earth in the center of a force that produces gravity) or whether it is some other cause which is not apparent. This concept is purely mathematical, for I am not now considering the physical causes and sites of forces.

Therefore, accelerative force is to motive force as velocity to motion. For quantity of motion arises from velocity and quantity of matter jointly, and motive force from accelerative force and quantity of matter jointly. For the sum of the actions of the accelerative force on the individual particles of

a body is the motive force of the whole body. As a consequence, near the surface of the earth, where the accelerative gravity, or the force that produces gravity, is the same in all bodies universally, the motive gravity, or weight, is as the body, but in an ascent to regions where the accelerative gravity becomes less, the weight will decrease proportionately and will always be as the body and the accelerative gravity jointly. Thus, in regions where the accelerative gravity is half as great, a body one-half or one-third as great will have a weight four or six times less.

Further, it is in this same sense that I call attractions and impulses accelerative and motive. Moreover, I use interchangeably and indiscriminately words signifying attraction, impulse, or any sort of propensity toward a center, considering these forces not from a physical but only from a mathematical point of view. Therefore, let the reader beware of thinking that by words of this kind I am anywhere defining a species or mode of action or a physical cause or reason, or that I am attributing forces in a true and physical sense to centers (which are mathematical points) if I happen to say that centers attract or that centers have forces.

Scholium Thus far it has seemed best to explain the senses in which less familiar words are to be taken in this treatise. Although time, space, place, and motion are very familiar to everyone, it must be noted that these quantities are popularly conceived solely with reference to the objects of sense perception. And this is the source of certain preconceptions; to eliminate them it is useful to distinguish these quantities into absolute and relative, true and apparent, mathematical and common.

1. Absolute, true, and mathematical time, in and of itself and of its own nature, without reference to anything external, flows uniformly and by another name is called duration. Relative, apparent, and common time is any sensible and external measure ^a(precise or imprecise)^a of duration by means of motion; such a measure—for example, an hour, a day, a month, a year—is commonly used instead of true time.

2. Absolute space, of its own nature without reference to anything external, always remains homogeneous and immovable. Relative space is any

aa. Newton uses the phrase "seu accuratâ seu inaequalis"—literally, "exact or nonuniform."

movable measure or dimension of this absolute space; such a measure or dimension is determined by our senses from the situation of the space with respect to bodies and is popularly used for immovable space, as in the case of space under the earth or in the air or in the heavens, where the dimension is determined from the situation of the space with respect to the earth. Absolute and relative space are the same in species and in magnitude, but they do not always remain the same numerically. For example, if the earth moves, the space of our air, which in a relative sense and with respect to the earth always remains the same, will now be one part of the absolute space into which the air passes, now another part of it, and thus will be changing continually in an absolute sense.

3. Place is the part of space that a body occupies, and it is, depending on the space, either absolute or relative. I say the part of space, not the position of the body or its outer surface. For the places of equal solids are always equal, while their surfaces are for the most part unequal because of the dissimilarity of shapes; and positions, properly speaking, do not have quantity and are not so much places as attributes of places. The motion of a whole is the same as the sum of the motions of the parts; that is, the change in position of a whole from its place is the same as the sum of the changes in position of its parts from their places, and thus the place of a whole is the same as the sum of the places of the parts and therefore is internal and in the whole body.

4. Absolute motion is the change of position of a body from one absolute place to another; relative motion is change of position from one relative place to another. Thus, in a ship under sail, the relative place of a body is that region of the ship in which the body happens to be or that part of the whole interior of the ship which the body fills and which accordingly moves along with the ship, and relative rest is the continuance of the body in that same region of the ship or same part of its interior. But true rest is the continuance of a body in the same part of that unmoving space in which the ship itself, along with its interior and all its contents, is moving. Therefore, if the earth is truly at rest, a body that is relatively at rest on a ship will move truly and absolutely with the velocity with which the ship is moving on the earth. But if the earth is also moving, the true and absolute motion of the body will arise partly from the true motion of the earth in unmoving space and partly from the relative motion of the ship on the earth. Further, if the body is also moving relatively on the ship, its true motion will arise partly from

the true motion of the earth in unmoving space and partly from the relative motions both of the ship on the earth and of the body on the ship, and from these relative motions the relative motion of the body on the earth will arise. For example, if that part of the earth where the ship happens to be is truly moving eastward with a velocity of 10,010 units, and the ship is being borne westward by sails and wind with a velocity of 10 units, and a sailor is walking on the ship toward the east with a velocity of 1 unit, then the sailor will be moving truly and absolutely in unmoving space toward the east with a velocity of 10,001 units and relatively on the earth toward the west with a velocity of 9 units.

In astronomy, absolute time is distinguished from relative time by the equation of common time. For natural days, which are commonly considered equal for the purpose of measuring time, are actually unequal. Astronomers correct this inequality in order to measure celestial motions on the basis of a truer time. It is possible that there is no uniform motion by which time may have an exact measure. All motions can be accelerated and retarded, but the flow of absolute time cannot be changed. The duration or perseverance of the existence of things is the same, whether their motions are rapid or slow or null; accordingly, duration is rightly distinguished from its sensible measures and is gathered from them by means of an astronomical equation. Moreover, the need for using this equation in determining when phenomena occur is proved by experience with a pendulum clock and also by eclipses of the satellites of Jupiter.

Just as the order of the parts of time is unchangeable, so, too, is the order of the parts of space. Let the parts of space move from their places, and they will move (so to speak) from themselves. For times and spaces are, as it were, the places of themselves and of all things. All things are placed in time with reference to order of succession and in space with reference to order of position. It is of the essence of spaces to be places, and for primary places to move is absurd. They are therefore absolute places, and it is only changes of position from these places that are absolute motions.

But since these parts of space cannot be seen and cannot be distinguished from one another by our senses, we use sensible measures in their stead. For we define all places on the basis of the positions and distances of things from some body that we regard as immovable, and then we reckon all motions with respect to these places, insofar as we conceive of bodies as being changed

in position with respect to them. Thus, instead of absolute places and motions we use relative ones, which is not inappropriate in ordinary human affairs, although in philosophy abstraction from the senses is required. For it is possible that there is no body truly at rest to which places and motions may be referred.

Moreover, absolute and relative rest and motion are distinguished from each other by their properties, causes, and effects. It is a property of rest that bodies truly at rest are at rest in relation to one another. And therefore, since it is possible that some body in the regions of the fixed stars or far beyond is absolutely at rest, and yet it cannot be known from the position of bodies in relation to one another in our regions whether or not any of these maintains a given position with relation to that distant body, true rest cannot be defined on the basis of the position of bodies in relation to one another.

It is a property of motion that parts which keep given positions in relation to wholes participate in the motions of such wholes. For all the parts of bodies revolving in orbit endeavor to recede from the axis of motion, and the impetus of bodies moving forward arises from the joint impetus of the individual parts. Therefore, when bodies containing others move, whatever is relatively at rest within them also moves. And thus true and absolute motion cannot be determined by means of change of position from the vicinity of bodies that are regarded as being at rest. For the exterior bodies ought to be regarded not only as being at rest but also as being truly at rest. Otherwise all contained bodies, besides being subject to change of position from the vicinity of the containing bodies, will participate in the true motions of the containing bodies and, if there is no such change of position, will not be truly at rest but only be regarded as being at rest. For containing bodies are to those inside them as the outer part of the whole to the inner part or as the shell to the kernel. And when the shell moves, the kernel also, without being changed in position from the vicinity of the shell, moves as a part of the whole.

A property akin to the preceding one is that when a place moves, whatever is placed in it moves along with it, and therefore a body moving away from a place that moves participates also in the motion of its place. Therefore, all motions away from places that move are only parts of whole and absolute motions, and every whole motion is compounded of the motion of a body away from its initial place, and the motion of this place away from

its place, and so on, until an unmoving place is reached, as in the above-mentioned example of the sailor. Thus, whole and absolute motions can be determined only by means of unmoving places, and therefore in what has preceded I have referred such motions to unmoving places and relative motions to movable places. Moreover, the only places that are unmoving are those that all keep given positions in relation to one another from infinity to infinity and therefore always remain immovable and constitute the space that I call immovable.

The causes which distinguish true motions from relative motions are the forces impressed upon bodies to generate motion. True motion is neither generated nor changed except by forces impressed upon the moving body itself, but relative motion can be generated and changed without the impression of forces upon this body. For the impression of forces solely on other bodies with which a given body has a relation is enough, when the other bodies yield, to produce a change in that relation which constitutes the relative rest or motion of this body. Again, true motion is always changed by forces impressed upon a moving body, but relative motion is not necessarily changed by such forces. For if the same forces are impressed upon a moving body and also upon other bodies with which it has a relation, in such a way that the relative position is maintained, the relation that constitutes the relative motion will also be maintained. Therefore, every relative motion can be changed while the true motion is preserved, and can be preserved while the true one is changed, and thus true motion certainly does not consist in relations of this sort.

The effects distinguishing absolute motion from relative motion are the forces of receding from the axis of circular motion. For in purely relative circular motion these forces are null, while in true and absolute circular motion they are larger or smaller in proportion to the quantity of motion. If a bucket is hanging from a very long cord and is continually turned around until the cord becomes twisted tight, and if the bucket is thereupon filled with water and is at rest along with the water and then, by some sudden force, is made to turn around in the opposite direction and, as the cord unwinds, perseveres for a while in this motion; then the surface of the water will at first be level, just as it was before the vessel began to move. But after the vessel, by the force gradually impressed upon the water, has caused the water also to begin revolving perceptibly, the water will gradually recede

from the middle and rise up the sides of the vessel, assuming a concave shape (as experience has shown me), and, with an ever faster motion, will rise further and further until, when it completes its revolutions in the same times as the vessel, it is relatively at rest in the vessel. The rise of the water reveals its endeavor to recede from the axis of motion, and from such an endeavor one can find out and measure the true and absolute circular motion of the water, which here is the direct opposite of its relative motion. In the beginning, when the relative motion of the water in the vessel was greatest, that motion was not giving rise to any endeavor to recede from the axis; the water did not seek the circumference by rising up the sides of the vessel but remained level, and therefore its true circular motion had not yet begun. But afterward, when the relative motion of the water decreased, its rise up the sides of the vessel revealed its endeavor to recede from the axis, and this endeavor showed the true circular motion of the water to be continually increasing and finally becoming greatest when the water was relatively at rest in the vessel. Therefore, that endeavor does not depend on the change of position of the water with respect to surrounding bodies, and thus true circular motion cannot be determined by means of such changes of position. The truly circular motion of each revolving body is unique, corresponding to a unique endeavor as its proper and sufficient effect, while relative motions are innumerable in accordance with their varied relations to external bodies and, like relations, are completely lacking in true effects except insofar as they participate in that true and unique motion. Thus, even in the system of those who hold that our heavens revolve below the heavens of the fixed stars and carry the planets around with them, the individual parts of the heavens, and the planets that are relatively at rest in the heavens to which they belong, are truly in motion. For they change their positions relative to one another (which is not the case with things that are truly at rest), and as they are carried around together with the heavens, they participate in the motions of the heavens and, being parts of revolving wholes, endeavor to recede from the axes of those wholes.

Relative quantities, therefore, are not the actual quantities whose names they bear but are those sensible measures of them (whether true or erroneous) that are commonly used instead of the quantities being measured. But if the meanings of words are to be defined by usage, then it is these sensible measures which should properly be understood by the terms "time,"

"space," "place," and "motion," and the manner of expression will be out of the ordinary and purely mathematical if the quantities being measured are understood here. Accordingly those who there interpret these words as referring to the quantities being measured do violence to the Scriptures. And they no less corrupt mathematics and philosophy who confuse true quantities with their relations and common measures.

It is certainly very difficult to find out the true motions of individual bodies and actually to differentiate them from apparent motions, because the parts of that immovable space in which the bodies truly move make no impression on the senses. Nevertheless, the case is not utterly hopeless. For it is possible to draw evidence partly from apparent motions, which are the differences between the true motions, and partly from the forces that are the causes and effects of the true motions. For example, if two balls, at a given distance from each other with a cord connecting them, were revolving about a common center of gravity, the endeavor of the balls to recede from the axis of motion could be known from the tension of the cord, and thus the quantity of circular motion could be computed. Then, if any equal forces were simultaneously impressed upon the alternate faces of the balls to increase or decrease their circular motion, the increase or decrease of the motion could be known from the increased or decreased tension of the cord, and thus, finally, it could be discovered which faces of the balls the forces would have to be impressed upon for a maximum increase in the motion, that is, which were the posterior faces, or the ones that are in the rear in a circular motion. Further, once the faces that follow and the opposite faces that precede were known, the direction of the motion would be known. In this way both the quantity and the direction of this circular motion could be found in any immense vacuum, where nothing external and sensible existed with which the balls could be compared. Now if some distant bodies were set in that space and maintained given positions with respect to one another, as the fixed stars do in the regions of the heavens, it could not, of course, be known from the relative change of position of the balls among the bodies whether the motion was to be attributed to the bodies or to the balls. But if the cord was examined and its tension was discovered to be the very one which the motion of the balls required, it would be valid to conclude that the motion belonged to the balls and that the bodies were at rest, and then, finally, from the change of position of the balls among the bodies, to determine

the direction of this motion. But in what follows, a fuller explanation will be given of how to determine true motions from their causes, effects, and apparent differences, and, conversely, of how to determine from motions, whether true or apparent, their causes and effects. For this was the purpose for which I composed the following treatise.

AXIOMS, OR THE LAWS OF MOTION



Law 1 *Every body perseveres in its state of being at rest or of moving ^auniformly straight forward,^a except insofar as ^bit^b is compelled to change ^cits^c state by forces impressed.*

Projectiles persevere in their motions, except insofar as they are retarded by the resistance of the air and are impelled downward by the force of gravity. A spinning hoop,^d which has parts that by their cohesion continually draw one another back from rectilinear motions, does not cease to rotate, except insofar as it is retarded by the air. And larger bodies—planets and comets—preserve for a longer time both their progressive and their circular motions, which take place in spaces having less resistance.

Law 2 *A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed.*

If some force generates any motion, twice the force will generate twice the motion, and three times the force will generate three times the motion, whether the force is impressed all at once or successively by degrees. And if the body was previously moving, the new motion (since motion is always in the same direction as the generative force) is added to the original motion if that motion was in the same direction or is subtracted from the original motion if it was in the opposite direction or, if it was in an oblique direction,

aa. See note bb to def. 3.

bb. Ed. 1 and ed. 2 lack the pronoun "illud," which, by expressing the subject, renders it somewhat more emphatic than it is when conveyed only by the form of the verb ("is compelled") and which makes more explicit the reference to an antecedent noun ("body").

cc. Ed. 1 and ed. 2 have "that."

d. The Latin word is "trochus," i.e., a top or some kind of spinner.

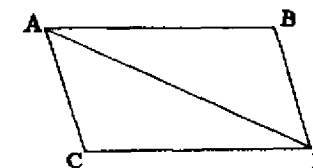
is combined obliquely and compounded with it according to the directions of both motions.

To any action there is always an opposite and equal reaction; in other words, the actions of two bodies upon each other are always equal and always opposite in direction. **Law 3**

Whatever presses or draws something else is pressed or drawn just as much by it. If anyone presses a stone with a finger, the finger is also pressed by the stone. If a horse draws a stone tied to a rope, the horse will (so to speak) also be drawn back equally toward the stone, for the rope, stretched out at both ends, will urge the horse toward the stone, and the stone toward the horse by one and the same endeavor to go slack and will impede the forward motion of the one as much as it promotes the forward motion of the other. If some body impinging upon another body changes the motion of that body in any way by its own force, then, by the force of the other body (because of the equality of their mutual pressure), it also will in turn undergo the same change in its own motion in the opposite direction. By means of these actions, equal changes occur in the motions, not in the velocities—that is, of course, if the bodies are not impeded by anything else.^a For the changes in velocities that likewise occur in opposite directions are inversely proportional to the bodies because the motions are changed equally. This law is valid also for attractions, as will be proved in the next scholium.

A body acted on by [two] forces acting jointly describes the diagonal of a parallelogram in the same time in which it would describe the sides if the forces were acting separately. **Corollary 1**

Let a body in a given time, by force M alone impressed in A, be carried with uniform motion from A to B, and, by force N alone impressed in the same place, be carried from A to C; then complete the parallelogram ABDC, and by both forces the body will be carried in the same time along the diagonal from A to D. For, since force N acts along the line AC parallel to



a. By "body" Newton means quantity of matter or mass (def. 1) and by "motion" he means quantity of motion (def. 2) or momentum.

mechanics—demonstrated in different ways by those who have written on this subject—depends on what has just now been said. For from this are easily derived the forces of machines, which are generally composed of wheels, drums, pulleys, levers, stretched strings, and weights, ascending directly or obliquely, and the other mechanical powers, as well as the forces of tendons to move the bones of animals.

Corollary 3 *The quantity of motion, which is determined by adding the motions made in one direction and subtracting the motions made in the opposite direction, is not changed by the action of bodies on one another.*

For an action and the reaction opposite to it are equal by law 3, and thus by law 2 the changes which they produce in motions are equal and in opposite directions. Therefore, if motions are in the same direction, whatever is added to the motion of the first body [*lit.* the fleeing body] will be subtracted from the motion of the second body [*lit.* the pursuing body] in such a way that the sum remains the same as before. But if the bodies meet head-on, the quantity subtracted from each of the motions will be the same, and thus the difference of the motions made in opposite directions will remain the same.

For example, suppose a spherical body A is three times as large as a spherical body B and has two parts of velocity, and let B follow A in the same straight line with ten parts of velocity; then the motion of A is to the motion of B as six to ten. Suppose that their motions are of six parts and ten parts respectively; the sum will be sixteen parts. When the bodies collide, therefore, if body A gains three or four or five parts of motion, body B will lose just as many parts of motion and thus after reflection body A will continue with nine or ten or eleven parts of motion and B with seven or six or five parts of motion, the sum being always, as originally, sixteen parts of motion. Suppose body A gains nine or ten or eleven or twelve parts of motion and so moves forward with fifteen or sixteen or seventeen or eighteen parts of motion after meeting body B; then body B, by losing as many parts of motion as A gains, will either move forward with one part, having lost nine parts of motion, or will be at rest, having lost its forward motion of ten parts, or will move backward with one part of motion, having lost its motion and (if I may say so) one part more, or will move backward with two parts of motion because a forward motion of twelve parts has been subtracted. And thus the sums, $15 + 1$ or $16 + 0$, of the motions in the same direction and the

differences, $17 - 1$ and $18 - 2$, of the motions in opposite directions will always be sixteen parts of motion, just as before the bodies met and were reflected. And since the motions with which the bodies will continue to move after reflection are known, the velocity of each will be found, on the supposition that it is to the velocity before reflection as the motion after reflection is to the motion before reflection. For example, in the last case, where the motion of body A was six parts before reflection and eighteen parts afterward, and its velocity was two parts before reflection, its velocity will be found to be six parts after reflection on the basis of the following statement: as six parts of motion before reflection is to eighteen parts of motion afterward, so two parts of velocity before reflection is to six parts of velocity afterward.

But if bodies that either are not spherical or are moving in different straight lines strike against each other obliquely and it is required to find their motions after reflection, the position of the plane by which the colliding bodies are touched at the point of collision must be determined; then (by corol. 2) the motion of each body must be resolved into two motions, one perpendicular to this plane and the other parallel to it. Because the bodies act upon each other along a line perpendicular to this plane, the parallel motions [*i.e.*, components] must be kept the same after reflection; and equal changes in opposite directions must be attributed to the perpendicular motions in such a way that the sum of the motions in the same direction and the difference of the motions in opposite directions remain the same as before the bodies came together. The circular motions of bodies about their own centers also generally arise from reflections of this sort. But I do not consider such cases in what follows, and it would be too tedious to demonstrate everything relating to this subject.

The common center of gravity of two or more bodies does not change its state whether of motion or of rest as a result of the actions of the bodies upon one another; and therefore the common center of gravity of all bodies acting upon one another (excluding external actions and impediments) either is at rest or moves uniformly straight forward. **Corollary 4**

For if two points move forward with uniform motion in straight lines, and the distance between them is divided in a given ratio, the dividing point either is at rest or moves forward uniformly in a straight line. This is demonstrated below in lem. 23 and its corollary for the case in which the motions

of the points take place in the same plane, and it can be demonstrated by the same reasoning for the case in which those motions do not take place in the same plane. Therefore, if any number of bodies move uniformly in straight lines, the common center of gravity of any two either is at rest or moves forward uniformly in a straight line, because any line joining these bodies through their centers—which move forward uniformly in straight lines—is divided by this common center in a given ratio. Similarly, the common center of gravity of these two bodies and any third body either is at rest or moves forward uniformly in a straight line, because the distance between the common center of the two bodies and the center of the third body is divided in a given ratio by the common center of the three. In the same way, the common center of these three and of any fourth body either is at rest or moves forward uniformly in a straight line, because that common center divides in a given ratio the distance between the common center of the three and the center of the fourth body, and so on without end. Therefore, in a system of bodies in which the bodies are entirely free of actions upon one another and of all other actions impressed upon them externally, and in which each body accordingly moves uniformly in its individual straight line, the common center of gravity of them all either is at rest or moves uniformly straight forward.

Further, in a system of two bodies acting on each other, since the distances of their centers from the common center of gravity are inversely as the bodies, the relative motions of these bodies, whether of approaching that center or of receding from it, will be equal. Accordingly, as a result of equal changes in opposite directions in the motions of these bodies, and consequently as a result of the actions of the bodies on each other, that center is neither accelerated nor retarded nor does it undergo any change in its state of motion or of rest. In a system of several bodies, the common center of gravity of any two acting upon each other does not in any way change its state as a result of that action, and the common center of gravity of the rest of the bodies (with which that action has nothing to do) is not affected by that action; the distance between these two centers is divided by the common center of gravity of all the bodies into parts inversely proportional to the total sums of the bodies whose centers they are, and (since those two centers maintain their state of moving or of being at rest) the common center of all maintains its state also—for all these reasons it is obvious that this common center of all never changes its state with respect to motion and rest as a result of the actions of two bodies upon

each other. Moreover, in such a system all the actions of bodies upon one another either occur between two bodies or are compounded of such actions between two bodies and therefore never introduce any change in the state of motion or of rest of the common center of all. Thus, since that center either is at rest or moves forward uniformly in some straight line, when the bodies do not act upon one another, that center will, notwithstanding the actions of the bodies upon one another, continue either to be always at rest or to move always uniformly straight forward, unless it is driven from this state by forces impressed on the system from outside. Therefore, the law is the same for a system of several bodies as for a single body with respect to perseverance in a state of motion or of rest. For the progressive motion, whether of a single body or of a system of bodies, should always be reckoned by the motion of the center of gravity.

When bodies are enclosed in a given space, their motions in relation to one another are the same whether the space is at rest or whether it is moving uniformly straight forward without circular motion. Corollary 5

For in either case the differences of the motions tending in the same direction and the sums of those tending in opposite directions are the same at the beginning (by hypothesis), and from these sums or differences there arise the collisions and impulses [*lit.* impetuses] with which the bodies strike one another. Therefore, by law 2, the effects of the collisions will be equal in both cases, and thus the motions with respect to one another in the one case will remain equal to the motions with respect to one another in the other case. This is proved clearly by experience: on a ship, all the motions are the same with respect to one another whether the ship is at rest or is moving uniformly straight forward.

If bodies are moving in any way whatsoever with respect to one another and are urged by equal accelerative forces along parallel lines, they will all continue to move with respect to one another in the same way as they would if they were not acted on by those forces. Corollary 6

For those forces, by acting equally (in proportion to the quantities of the bodies to be moved) and along parallel lines, will (by law 2) move all the bodies equally (with respect to velocity), and so will never change their positions and motions with respect to one another.

Scholium The principles I have set forth are accepted by mathematicians and confirmed by experiments of many kinds. By means of the first two laws and the first two corollaries Galileo found that the descent of heavy bodies is in the squared ratio of the time and that the motion of projectiles occurs in a parabola, as experiment confirms, except insofar as these motions are somewhat retarded by the resistance of the air. ^aWhen a body falls, uniform gravity, by acting equally in individual equal particles of time, impresses equal forces upon that body and generates equal velocities; and in the total time it impresses a total force and generates a total velocity proportional to the time. And the spaces described in proportional times are as the velocities and the times jointly, that is, in the squared ratio of the times. And when a body is projected upward, uniform gravity impresses forces and takes away velocities proportional to the times; and the times of ascending to the greatest heights are as the velocities to be taken away, and these heights are as the velocities and the times jointly, or as the squares of the velocities. And when a body is projected along any straight line, its motion arising from the projection is compounded with the motion arising from gravity.

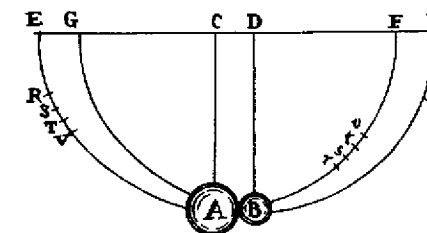
For example, let body A by the motion of projection alone describe the straight line AB in a given time, and by the motion of falling alone describe the vertical distance AC in the same time; then complete the parallelogram ABDC, and by the compounded motion the body will be found in place D at the end of the time; and the curved line AED which the body will describe will be a parabola which the straight line AB touches at A and whose ordinate BD is as AB^2 .^a

What has been demonstrated concerning the times of oscillating pendulums depends on the same first two laws and first two corollaries, and this is supported by daily experience with clocks. From the same laws and corollaries and law 3, Sir Christopher Wren, Dr. John Wallis, and Mr. Christiaan Huygens, easily the foremost geometers of the previous generation, independently found the rules of the collisions and reflections of hard bodies, and communicated them to the Royal Society at nearly the same time, entirely agreeing with one another (as to these rules); and Wallis was indeed the

aa. Ed. 1 and ed. 2 lack this.

first to publish what had been found, followed by Wren and Huygens. But Wren additionally proved the truth of these rules before the Royal Society by means of an experiment with pendulums, which the eminent Mariotte soon after thought worthy to be made the subject of a whole book.

However, if this experiment is to agree precisely with the theories, account must be taken of both the resistance of the air and the elastic force of the colliding bodies. Let the spherical bodies A and B be suspended from centers C and D by parallel and equal cords AC and BD. With these centers and with those distances as radii describe semicircles EAF and GBH bisected by radii CA and DB. Take away body B, and let body A be brought to any point R of the arc



EAF and be let go from there, and let it return after one oscillation to point V. RV is the retardation arising from the resistance of the air. Let ST be a fourth of RV and be located in the middle so that RS and TV are equal and RS is to ST as 3 to 2. Then ST will closely approximate the retardation in the descent from S to A. Restore body B to its original place. Let body A fall from point S, and its velocity at the place of reflection A, without sensible error, will be as great as if it had fallen in a vacuum from place T. Therefore let this velocity be represented by the chord of the arc TA. For it is a proposition very well known to geometers that the velocity of a pendulum in its lowest point is as the chord of the arc that it has described in falling. After reflection let body A arrive at place *s*, and body B at place *k*. Take away body B and find place *v* such that if body A is let go from this place and after one oscillation returns to place *r*, *st* will be a fourth of *rv* and be located in the middle, so that *rs* and *tv* are equal; and let the chord of the arc *tA* represent the velocity that body A had in place A immediately after reflection. For *t* will be that true and correct place to which body A must have ascended if there had been no resistance of the air. By a similar method the place *k*, to which body B ascends, will have to be corrected, and the place *l*, to which that body must have ascended in a vacuum, will have to be found. In this manner it is possible to make all our experiments, just as if we were in a vacuum. Finally body A will have to be multiplied (so to speak) by the chord of the arc TA, which represents its velocity, in order

to get its motion in place A immediately before reflection, and then by the chord of the arc tA in order to get its motion in place A immediately after reflection. And thus body B will have to be multiplied by the chord of the arc Bl in order to get its motion immediately after reflection. And by a similar method, when two bodies are let go simultaneously from different places, the motions of both will have to be found before as well as after reflection, and then finally the motions will have to be compared with each other in order to determine the effects of the reflection.

On making a test in this way with ten-foot pendulums, using unequal as well as equal bodies, and making the bodies come together from very large distances apart, say of eight or twelve or sixteen feet, I always found—within an error of less than three inches in the measurements—that when the bodies met each other directly, the changes of motions made in the bodies in opposite directions were equal, and consequently that the action and reaction were always equal. For example, if body A collided with body B, which was at rest, with nine parts of motion and, losing seven parts, proceeded after reflection with two, body B rebounded with those seven parts. If the bodies met head-on, A with twelve parts of motion and B with six, and A rebounded with two, B rebounded with eight, fourteen parts being subtracted from each. Subtract twelve parts from the motion of A and nothing will remain; subtract another two parts, and a motion of two parts in the opposite direction will be produced; and so, subtracting fourteen parts from the six parts of the motion of body B, eight parts will be produced in the opposite direction. But if the bodies moved in the same direction, A more quickly with fourteen parts and B more slowly with five parts, and after reflection A moved with five parts, then B moved with fourteen, nine parts having been transferred from A to B. And so in all other cases. As a result of the meeting and collision of bodies, the quantity of motion—determined by adding the motions in the same direction and subtracting the motions in opposite directions—was never changed. I would attribute the error of an inch or two in the measurements to the difficulty of doing everything with sufficient accuracy. It was difficult both to release the pendulums simultaneously in such a way that the bodies would impinge upon each other in the lowest place AB, and to note the places s and k to which the bodies ascended after colliding. But also, with respect to the pendulous bodies themselves, errors were introduced by the unequal density of the parts and by irregularities of texture arising from other causes.

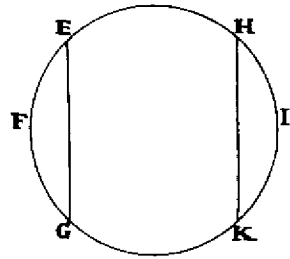
Further, lest anyone object that the rule which this experiment was designed to prove presupposes that bodies are either absolutely hard or at least perfectly elastic and thus of a kind which do not occur ^bnaturally,^b I add that the experiments just described work equally well with soft bodies and with hard ones, since surely they do not in any way depend on the condition of hardness. For if this rule is to be tested in bodies that are not perfectly hard, it will only be necessary to decrease the reflection in a fixed proportion to the quantity of elastic force. In the theory of Wren and Huygens, absolutely hard bodies rebound from each other with the velocity with which they have collided. This will be affirmed with more certainty of perfectly elastic bodies. In imperfectly elastic bodies the velocity of rebounding must be decreased together with the elastic force, because that force (except when the parts of the bodies are damaged as a result of collision, or experience some sort of extension such as would be caused by a hammer blow) is fixed and determinate (as far as I can tell) and makes the bodies rebound from each other with a relative velocity that is in a given ratio to the relative velocity with which they collide. I have tested this as follows with tightly wound balls of wool strongly compressed. First, releasing the pendulums and measuring their reflection, I found the quantity of their elastic force; then from this force I determined what the reflections would be in other cases of their collision, and the experiments which were made agreed with the computations. The balls always rebounded from each other with a relative velocity that was to the relative velocity of their colliding as 5 to 9, more or less. Steel balls rebounded with nearly the same velocity and cork balls with a slightly smaller velocity, while with glass balls the proportion was roughly 15 to 16. And in this manner the third law of motion—insofar as it relates to impacts and reflections—is proved by this theory, which plainly agrees with experiments.

I demonstrate the third law of motion for attractions briefly as follows. Suppose that between any two bodies A and B that attract each other any obstacle is interposed so as to impede their coming together. If one body A is more attracted toward the other body B than that other body B is attracted toward the first body A, then the obstacle will be more strongly pressed by body A than by body B and accordingly will not remain in equilibrium. The stronger pressure will prevail and will make the system of the two bodies and

bb. Evidently "in natural compositions" or "in natural bodies."

the obstacle move straight forward in the direction from A toward B and, in empty space, go on indefinitely with a motion that is always accelerated, which is absurd and contrary to the first law of motion. For according to the first law, the system will have to persevere in its state of resting or of moving uniformly straight forward, and accordingly the bodies will urge the obstacle equally and on that account will be equally attracted to each other. I have tested this with a lodestone and iron. If these are placed in separate vessels that touch each other and float side by side in still water, neither one will drive the other forward, but because of the equality of the attraction in both directions they will sustain their mutual endeavors toward each other, and at last, having attained equilibrium, they will be at rest.

“In the same way gravity is mutual between the earth and its parts. Let the earth FI be cut by any plane EG into two parts EGF and EGI; then their



weights toward each other will be equal. For if the greater part EGI is cut into two parts EGKH and HKI by another plane HK parallel to the first plane EG, in such a way that HKI is equal to the part EFG that has been cut off earlier, it is manifest that the middle part EGKH will not preponderate toward either of the outer parts but will, so to speak, be suspended in equilibrium

between both and will be at rest. Moreover, the outer part HKI will press upon the middle part with all its weight and will urge it toward the other outer part EGF, and therefore the force by which EGI, the sum of the parts HKI and EGKH, tends toward the third part EGF is equal to the weight of the part HKI, that is, equal to the weight of the third part EGF. And therefore the weights of the two parts EGI and EGF toward each other are equal, as I set out to demonstrate. And if these weights were not equal, the whole earth, floating in an aether free of resistance, would yield to the greater weight and in receding from it would go off indefinitely.^c

As bodies are equipollent in collisions and reflections if their velocities are inversely as their inherent forces [i.e., forces of inertia], so in the motions of machines those agents [i.e., acting bodies] whose velocities (reckoned in the direction of their forces) are inversely as their inherent forces are equipol-

cc. Ed. 1 lacks this.

lent and sustain one another by their contrary endeavors. Thus weights are equipollent in moving the arms of a balance if during oscillation of the balance they are inversely as their velocities upward and downward; that is, weights which move straight up and down are equipollent if they are inversely as the distances between the axis of the balance and the points from which they are suspended; but if such weights are interfered with by oblique planes or other obstacles that are introduced and thus ascend or descend obliquely, they are equipollent if they are inversely as the ascents and descents insofar as these are reckoned with respect to a perpendicular, and this is so because the direction of gravity is downward. Similarly, in a pulley or combination of pulleys, the weight will be sustained by the force of the hand pulling the rope vertically, which is to the weight (ascending either straight up or obliquely) as the velocity of the perpendicular ascent to the velocity of the hand pulling the rope. In clocks and similar devices, which are constructed out of engaged gears, the contrary forces that promote and hinder the motion of the gears will sustain each other if they are inversely as the velocities of the parts of the gears upon which they are impressed. The force of a screw to press a body is to the force of a hand turning the handle as the circular velocity of the handle, in the part where it is urged by the hand, is to the progressive velocity of the screw toward the pressed body. The forces by which a wedge presses the two parts of the wood that it splits are to the force of the hammer upon the wedge as the progress of the wedge (in the direction of the force impressed upon it by the hammer) is to the velocity with which the parts of the wood yield to the wedge along lines perpendicular to the faces of the wedge. And the case is the same for all machines.

The effectiveness and usefulness of all machines or devices consist wholly in our being able to increase the force by decreasing the velocity, and vice versa; in this way the problem is solved in the case of any working machine or device: “To move a given weight by a given force” or to overcome any other given resistance by a given force. For if machines are constructed in such a way that the velocities of the agent [or acting body] and the resistant [or resisting body] are inversely as the forces, the agent will sustain the resistance and, if there is a greater disparity of velocities, will overcome that resistance. Of course the disparity of the velocities may be so great that it can also overcome all the resistance which generally arises from the friction of contiguous bodies sliding over one another, from the cohesion of continuous

bodies that are to be separated from one another, or from the weights of bodies to be raised; and if all this resistance is overcome, the remaining force will produce an acceleration of motion proportional to itself, partly in the parts of the machine, partly in the resisting body.^d

But my purpose here is not to write a treatise on mechanics. By these examples I wished only to show the wide range and the certainty of the third law of motion. For if the action of an agent is reckoned by its force and velocity jointly, and if, similarly, the reaction of a resistant is reckoned jointly by the velocities of its individual parts and the forces of resistance arising from their friction, cohesion, weight, and acceleration, the action and reaction will always be equal to each other in all examples of using devices or machines. And to the extent to which the action is propagated through the machine and ultimately impressed upon each resisting body, its ultimate direction will always be opposite to the direction of the reaction.

d. Newton writes of "instrumentorum" (literally, "equipment") and of "instrumentis mechanicis" (literally, "mechanical instruments"), as well as "machinae." See §5.7 of the Guide.

BOOK 1

THE MOTION OF BODIES

